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Lainiotis filter, golden section and Fibonacci sequence

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ABSTRACT

The relation between the discrete time Lainiotis filter on the one side and the golden section and the Fibonacci sequence on the other is established. As far as the random walk system is concerned, the relation between the Lainiotis filter and the golden section is derived through the Riccati equation since the steady state estimation error covariance is related to the golden section. The relation between the closed form of the Lainiotis filter and the Fibonacci sequence is also derived. It is shown that the steady state Lainiotis filter computes the state estimate using a linear combination of the previous estimate and of the current measurement with coefficients related to the golden section. A Finite Impulse Response (FIR) implementation of the steady state Lainiotis filter is also proposed, where the filter computes the state estimate as a linear combination of a well-defined set of the last measurements with coefficients which are powers of the golden section. Finally, the scalar generic stochastic dynamic system is considered and the relation between its parameters and the golden section is investigated.

1. Introduction

Although the connection between the golden section and Fibonacci numbers with nature, arts and architecture is known for centuries, there is presently a huge interest of modern sciences in these classical theories. Particularly, researchers in the computer science (measurement theory and communication systems [14]) and cryptography [12] exhibit a substantial interest in these classical theories and use them in order to model phenomena in their field. The above are only a few applications of the golden section and the Fibonacci numbers that imply a new mathematical direction which is the creation of a fascinating and beautiful subject of the "Mathematics of Harmony" [13].

Filtering plays an important role in many fields of science: applications to aerospace industry, chemical process, communication systems design, control, civil engineering, filtering noise from 2-dimensional images, pollution prediction and power systems are mentioned in [1]. In the field of signal processing, measurements are available containing the signal and the noise and the task is to produce an estimate of the signal through processing of the measurements by a filter. In this area the discrete time Kalman filter [1,8] and Lainiotis filter [9,10] are well-known algorithms that solve the filtering problem. Lainiotis filter uses the "partitioning approach" to estimation leading to robust, computationally effective and fast filtering algorithms [10]. The two filters are equivalent to each other [3] since they compute theoretically the same estimations. A key difference between the two filters is the fact that Kalman filter computes the estimation through prediction, while Lainiotis filter computes the estimation through smoothing. Another difference is that in the Kalman filter case the initial state has a

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Gaussian probability density function (pdf), while in the Lainiotis filter case the pdf could be also non-Gaussian [11]. Which time invariant filter is faster depends on the relation between the state and measurement dimensions: in fact Kalman filter is faster than Lainiotis filter in multi-state problems (the state dimension is enough greater than the measurement dimension), while Lainiotis filter is faster than Kalman filter in the multi-sensor problems (the measurement dimension is enough greater than the state dimension), as for instance the multi-sensor systems with many sensing devices are considered in the seismic processing, see [3].

Recently, the relation between the discrete time Kalman filter and the golden section is established [4,5]. The novelty of the paper is to establish the relation between the discrete time Lainiotis filter on the one side and the golden section and the Fibonacci sequence on the other. In fact, the relation between the discrete time Lainiotis filter and the golden section is described for the scalar generic stochastic dynamic system; similar relation has been described for the Kalman filter assuming scalar systems with special output coefficient (equal to one). It is also shown that the prediction/estimation/smoothing error covariances are related to the golden section. Finally, a FIR implementation of the steady state Lainiotis filter is proposed, where the filter coefficients are powers of the golden section.

The paper is organized as follows: In Sections 2 and 3 a brief review of the golden section and the Fibonacci sequence, respectively, is presented while in Section 4 the discrete time Lainiotis filter is presented. In Section 5 the random walk system is considered. The relation between the discrete time Lainiotis filter and the golden section is established through the Riccati equation. Also, the relation between the closed form of the Lainiotis filter and the Fibonacci sequence is derived. Moreover, the relation between the steady state Lainiotis filter and the golden section is described. In addition a Finite Impulse Response (FIR) implementation of the steady state Lainiotis filter is proposed, where the coefficients of the filter are related to the golden section. In Section 6 the relation between the Lainiotis filter, the Kalman filter and the golden section is described, for the random walk system. In Section 7 the scalar generic stochastic dynamic system is considered and the relation between the system parameters and the golden section is investigated. Finally, Section 8 summarizes the conclusions.

2. Golden section

The ‘‘Golden Ratio’’ as a concept has a long history in mathematics. Although the term ‘‘golden section’’ appears in print for the first time by the German mathematician Martin Ohm, the younger brother of the well-known physicist George Simon Ohm, in a footnote in the 1835 second edition of Die Reine Elementar-Mathematik [6], the mathematical concept of Golden Ratio traces back to the famous Greek mathematician Euclid from Alexandria. Particularly, Euclid's Elements [7] (Greek: $\Sigma\tau o\iota\chi\epsilon\iota\alpha$) provide the first known written definition of what is now called the golden ratio in order to solve a geometrical problem concerning the division of a line segment in extreme and mean ratio. The definition is the following: ‘‘A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less’’. We provide below the essence of this geometrical problem: A line segment AB must be divided with a point C into two parts so that the ratio between the shorter part AC and the longer one CB is equal to the ratio between the longer part CB and the whole line segment AB, i.e.:

$$\lambda = \frac{AC}{CB} = \frac{CB}{AB}$$

Using the relationship $AB = AC + CB$ we take:

$$\lambda = \frac{AC}{CB} = \frac{CB}{AB} = \frac{CB}{AC+CB} = \frac{CB/CB}{AC/CB+CB/CB} = \frac{1}{\lambda+1}$$

Hence, the equation to calculate the ratio λ is

$$\lambda^2 + \lambda - 1 = 0 \tag{1}$$

with two roots:

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$$

The positive root of Eq. (1) is the so-called *golden section*:

$$\alpha = \frac{-1+\sqrt{5}}{2} \approx 0.618 \tag{2}$$

The golden section has the following property:

$$1-\alpha = \alpha^2 \tag{3}$$

The reciprocal of the golden section is the golden ratio. So, the well-known number ϕ (phi), the *golden ratio*, is given as follows:

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618 \tag{4}$$

The relations between the golden section and the golden ratio are derived by (2) and (4) and are given below:

$$\phi = \frac{1}{\alpha} = 1+\alpha \tag{5}$$

The golden section seems to appear in many of the proportions of famous ancient buildings, such as the Parthenon in Athens. Also the proportions of famous paintings seem to be designed according to the golden section, for example Botticelli's Venus in the painting La Primavera or Vergine delle Rocce created by Leonardo Da Vinci. However, there is no original documentary evidence that these buildings and paintings were deliberately designed using the golden section.

The golden ratio has been of interest to mathematicians, philosophers, architects but in the last decades computer scientists and engineers have also concentrated their work on this ratio. Recently, concerning the signal processing research area, the relation between the discrete time Kalman filter and the golden section is established [4,5].

3. Fibonacci sequence

Fibonacci, an Italian born mathematician, discovered his unique number sequence theory in around 1200 AD;

the Fibonacci sequence, where the seed values are f_1 and f_2 starting from two seeds values the $f_1=0$ and $f_2=1$, is defined by recurrence by taking each subsequent number as the sum of the two previous ones:

$$f_{v+2}=f_{v+1}+f_v, \quad v\geq 1, \quad \text{and} \quad f_1=0,\ f_2=1 \tag{6}$$

Thus, the sequence of Fibonacci numbers is {0,1,1, 2,3,5,8,13,21,34,55,89,144, ...,}. It is well-known that the Fibonacci sequence satisfies the limit properties [4]:

$$\lim_{v\to\infty}\frac{f_v}{f_{v+1}}=\alpha \quad \text{and} \quad \lim_{v\to\infty}\frac{f_{v+2}}{f_{v+1}}=\phi \tag{7}$$

which provide the relation between the Fibonacci sequence on the one side and the golden section and the golden ratio on the other side. Today, it is believed that Fibonacci himself did not even realize the connection of the Fibonacci sequence to the golden ratio.

Fibonacci numbers seem to appear also in nature. For example, many types of flowers have a Fibonacci number of petals: daisies tend to have 34 or 55 petals, sunflowers have 89 or 144. Similarly, the numbers of rings on the trunks of palm trees and the scales on the surface of a pineapple follow a sequence of Fibonacci numbers. Recently, concerning the signal processing research area, the relation between the discrete time Kalman filter and the Fibonacci sequence is described [4,5].

4. Lainiotis filter

Consider the time invariant stochastic dynamic system described by the following state space equations:

$$x(k+1)=Fx(k)+w(k)$$
$$z(k+1)=Hx(k+1)+v(k+1) \tag{8}$$

for $k=0,1,\ldots,$ where $x(k)$ is $n\times 1$ state vector at time k, $z(k)$ is $m\times 1$ measurement vector, F is $n\times n$ system transition matrix, H is $m\times n$ output matrix, $\{w(k)\},\{v(k)\}$ are independent Gaussian zero-mean white and uncorrelated random processes, Q is $n\times n$ plant noise covariance matrix, R is $m\times m$ measurement noise covariance matrix, and $x(0)$ is a Gaussian random process with mean x_0 and covariance P_0.

The filtering problem is to produce an estimate at time L of the state vector using measurements till time L, i.e. the aim is to use the measurements set $\{z(1), z(2), \ldots, z(L)\}$ in order to calculate an estimate value $x(L/L)$ of the state vector $x(L)$. The discrete time Lainiotis filter [9,10] is a well-known algorithm that solve the filtering problem, by computing the estimation $x(k/k)$ at time k, and the corresponding estimation error covariance matrix $P(k/k)$ such that

$$P(k+1/k+1)=P_n+F_n[I+P(k/k)O_n]^{-1}P(k/k)F_n^T \tag{9}$$

$$\begin{aligned}x(k+1/k+1)&=F_n[I+P(k/k)O_n]^{-1}x(k/k)\\&\quad+(K_n+F_n[I+P(k/k)O_n]^{-1}P(k/k)K_m)z(k+1)\end{aligned} \tag{10}$$

for $k=0,1,\ldots,$ with initial conditions $P(0/0)=P_0$, and $x(0/0)=x_0$, where the following constant matrices are calculated off-line:

$$A=[HQH^T+R]^{-1}$$
$$K_n=QH^TA$$
$$K_m=F^TH^TA$$
$$P_n=Q-QH^TAHQ$$
$$F_n=F-QH^TAHF$$
$$O_n=F^TH^TAHF \tag{11}$$

Recall that the covariance matrices Q,R and $P(k/k)$ are non-negative definite matrices; hereafter these matrices are considered to be positive definite. Then, the existence of the $m\times m$ symmetric matrix $A=[HQH^T+R]^{-1}$ is guaranteed, when R is a positive definite ($R>0$), which means that no measurement is exact. This is reasonable in physical problems. Moreover, the existence of $[I+P(k/k)O_n]^{-1}$ is guaranteed due to the presence of the identity matrix I and due to the facts that $P(k/k)>0$ and $O_n>0$, since $A>0$.

Note that if the signal process system is asymptotically stable (i.e. all the eigenvalues of F lie inside the unit circle), then there exist a unique positive definite steady state value P_e of the estimation error covariance matrix, i.e. the estimation error covariance $P(k/k)$ tends to the steady state estimation error covariance:

$$P_e=\lim_{k\to\infty}P(k/k) \tag{12}$$

This steady state solution P_e can be calculated by recursively implementing the *Riccati equation* emanating from Lainiotis filter (10) with the initial condition $P(0/0)=P_0$. The steady state or limiting solution of the Riccati equation is independent of the initial condition.

The steady state estimation error covariance matrix satisfies the *algebraic Riccati equation* emanating from Lainiotis filter:

$$P_e=P_n+F_n[I+P_eO_n]^{-1}P_eF_n^T \tag{13}$$

The steady state estimation error covariance matrix may exist even if the system is not asymptotically stable.

5. Lainiotis filter for the random walk system

Consider the *random walk system*, namely the scalar ($n=1$ and $m=1$) stochastic dynamic system in (8) with the transition and output coefficients equal to one, $F=f=1$ and $H=h=1$:

$$x(k+1)=x(k)+w(k)$$
$$z(k+1)=x(k+1)+v(k+1) \tag{14}$$

and the process and measurement noise sources having equal noise covariances

$$Q=q=\sigma^2, \quad R=r=\sigma^2. \tag{15}$$

Then the Lainiotis filter parameters by (11) are

$$A=\frac{1}{2\sigma^2}, \quad K_n=\frac{1}{2}, \quad K_m=\frac{1}{2\sigma^2},$$
$$P_n=\frac{1}{2}\sigma^2, \quad F_n=\frac{1}{2}, \quad O_n=\frac{1}{2\sigma^2} \tag{16}$$

In this section, we deal with the discrete time Lainiotis filter and show how is related to the golden section as well as to the Fibonacci sequence.

5.1. Lainiotis filter and the golden ratio

Substituting the values of parameters by (16) in (9) and (10) the recursive form of the Lainiotis filter is derived:

$$P(k+1/k+1)=\frac{\sigma^2+P(k/k)}{2\sigma^2+P(k/k)}\sigma^2 \tag{17}$$

$$x(k+1/k+1)=\frac{\sigma^2}{2\sigma^2+P(k/k)}x(k/k)+\frac{\sigma^2+P(k/k)}{2\sigma^2+P(k/k)}z(k+1) \tag{18}$$

for $k=0,1,\ldots$, with initial conditions $P(0/0)=P_0$, and $x(0/0)=x_0$.

Moreover, the steady state value P_e of the estimation error covariance can be calculated by solving the following algebraic Riccati equation, resulting from (13) by substituting the values of Lainiotis filter parameters by (16):

$$P_e^2+\sigma^2P_e-\sigma^4=0 \tag{19}$$

The unique positive solution of Eq. (19) is the steady state estimation error covariance P_e and is related to the golden section; in fact by (2) it is equal to the golden section times the covariance σ^2:

$$P_e=\frac{-1+\sqrt{5}}{2}\sigma^2=\alpha\sigma^2 \tag{20}$$

Moreover, we observe that the negative solution of (19) is related also to the golden ratio:

$$\frac{-1-\sqrt{5}}{2}\sigma^2=-\frac{1+\sqrt{5}}{2}\sigma^2=-\phi\sigma^2$$

Thus, it has been shown that for the random walk system, the steady state estimation error covariance of the corresponding Lainiotis filter is related to the golden section.

5.2. Lainiotis filter and the Fibonacci sequence

The recursive form (open form) of Lainiotis filter consists of the recursive equations (17) and (18). We are able to derive the non-recursive form (closed form) of Lainiotis filter from these equations. We are going to show that for the random walk system, the coefficients of the closed form of the Lainiotis filter are related to the Fibonacci numbers.

The recursive equations of the Lainiotis filter (17) and (18) can be written in the following *closed form* of the *Lainiotis filter*:

$$P(k+1/k+1)=\frac{f_{2k+3}\sigma^2+f_{2k+2}P(0/0)}{f_{2k+4}\sigma^2+f_{2k+3}P(0/0)}\sigma^2 \tag{21}$$

$$x(k+1/k+1)=\frac{1}{f_{2k+4}\sigma^2+f_{2k+3}P(0/0)}\times\left[\sigma^2x(0/0)+\sum_{i=1}^{k+1}(f_{2i+1}\sigma^2+f_{2i}P(0/0))z(i)\right] \tag{22}$$

for $k=0,1,\ldots$, with initial conditions $P(0/0)=P_0$, and $x(0/0)=x_0$.

Proof. The proof of (21) is based on the induction method. Using the definition of the Fibonacci sequence by (6) and the recursive equation (17) for $k=0$ we can write

$$P(1/1)=\frac{\sigma^2+P_0}{2\sigma^2+P_0}\sigma^2=\frac{f_3\sigma^2+f_2P_0}{f_4\sigma^2+f_3P_0}\sigma^2$$

which satisfies (21).

Assume that the formula in (21) is true for k, then by (17) we have

$$P(k+2/k+2)=\frac{\sigma^2+P(k+1/k+1)}{2\sigma^2+P(k+1/k+1)}\sigma^2$$

In the last equation, substituting the assumption of the induction and using the definition of Fibonacci sequence by (6) we derive

$$\begin{aligned}P(k+2/k+2)&=\frac{\sigma^2+\dfrac{f_{2k+3}\sigma^2+f_{2k+2}P_0}{f_{2k+4}\sigma^2+f_{2k+3}P_0}\sigma^2}{2\sigma^2+\dfrac{f_{2k+3}\sigma^2+f_{2k+2}P_0}{f_{2k+4}\sigma^2+f_{2k+3}P_0}\sigma^2}\sigma^2\\&=\frac{(f_{2k+3}+f_{2k+4})\sigma^2+(f_{2k+2}+f_{2k+3})P_0}{(2f_{2k+4}+f_{2k+3})\sigma^2+(2f_{2k+3}+f_{2k+2})P_0}\sigma^2\\&=\frac{f_{2k+5}\sigma^2+f_{2k+4}P_0}{(f_{2k+4}+(f_{2k+4}+f_{2k+3}))\sigma^2+(f_{2k+3}+(f_{2k+3}+f_{2k+2}))P_0}\sigma^2\\&=\frac{f_{2k+5}\sigma^2+f_{2k+4}P_0}{f_{2k+6}\sigma^2+f_{2k+5}P_0}\sigma^2\\&=\frac{f_{2(k+1)+3}\sigma^2+f_{2(k+1)+2}P_0}{f_{2(k+1)+4}\sigma^2+f_{2(k+1)+3}P_0}\sigma^2\end{aligned}$$

i.e. (21) holds also for $k+1$, which completes the induction method for (21).

The proof of (22) is also based on the induction method. In fact, for $k=0$ the recursive equation (18) yields

$$\begin{aligned}x(1/1)&=\frac{1}{2\sigma^2+P_0}\sigma^2x_0+\frac{\sigma^2+P_0}{2\sigma^2+P_0}z(1)\\&=\frac{1}{2\sigma^2+P_0}[\sigma^2x_0+(\sigma^2+P_0)z(1)]\\&=\frac{1}{f_4\sigma^2+f_3P_0}[\sigma^2x_0+(f_3\sigma^2+f_2P_0)z(1)]\end{aligned}$$

which satisfies (22).

Assume that the formula in (22) is true for k, then by (18) we have

$$x(k+2/k+2)=\frac{\sigma^2}{2\sigma^2+P(k+1/k+1)}x(k+1/k+1)+\frac{\sigma^2+P(k+1/k+1)}{2\sigma^2+P(k+1/k+1)}z(k+2)$$

In the last equation, substituting the assumption of the induction and using (21) and the definition of Fibonacci sequence by (6) we derive

$$
\begin{aligned}
x(k+2/k+2) &= \frac{f_{2k+4}\sigma^2+f_{2k+3}P_0}{(2f_{2k+4}+f_{2k+3})\sigma^2+(2f_{2k+3}+f_{2k+2})P_0} \\
&\times\left\{\frac{\sigma^2}{f_{2k+4}\sigma^2+f_{2k+3}P_0}x_0+\frac{1}{f_{2k+4}\sigma^2+f_{2k+3}P_0}\right. \\
&\left.\times\sum_{i=1}^{k+1}(f_{2i+1}\sigma^2+f_{2i}P_0)z(i)\right\} \\
&+\frac{f_{2k+5}\sigma^2+f_{2k+4}P_0}{f_{2k+6}\sigma^2+f_{2k+5}P_0}z(k+2) \\
&= \frac{\sigma^2}{(f_{2k+4}+(f_{2k+4}+f_{2k+3}))\sigma^2+(f_{2k+3}+(f_{2k+3}+f_{2k+2}))P_0}x_0 \\
&+\frac{1}{(f_{2k+4}+(f_{2k+4}+f_{2k+3}))\sigma^2+(f_{2k+3}+(f_{2k+3}+f_{2k+2}))P_0} \\
&\times\sum_{i=1}^{k+1}(f_{2i+1}\sigma^2+f_{2i}P_0)z(i)+\frac{f_{2k+5}\sigma^2+f_{2k+4}P_0}{f_{2k+6}\sigma^2+f_{2k+5}P_0}z(k+2) \\
&= \frac{\sigma^2}{f_{2k+6}\sigma^2+f_{2k+5}P_0}x_0+\frac{1}{f_{2k+6}\sigma^2+f_{2k+5}P_0} \\
&\times\sum_{i=1}^{k+1}(f_{2i+1}\sigma^2+f_{2i}P_0)z(i)+\frac{f_{2k+5}\sigma^2+f_{2k+4}P_0}{f_{2k+6}\sigma^2+f_{2k+5}P_0}z(k+2) \\
&= \frac{\sigma^2}{f_{2k+6}\sigma^2+f_{2k+5}P_0}x_0+\frac{1}{f_{2k+6}\sigma^2+f_{2k+5}P_0} \\
&\times\left[\sum_{i=1}^{k+1}(f_{2i+1}\sigma^2+f_{2i}P_0)z(i)+(f_{2k+5}\sigma^2+f_{2k+4}P_0)z(k+2)\right] \\
&= \frac{\sigma^2}{f_{2k+6}\sigma^2+f_{2k+5}P_0}x_0+\frac{1}{f_{2k+6}\sigma^2+f_{2k+5}P_0} \\
&\times\sum_{i=1}^{k+2}(f_{2i+1}\sigma^2+f_{2i}P_0)z(i)
\end{aligned}
$$

i.e. (22) holds also for $k+1$, which completes the induction method for (22). □

Thus, it is obvious that for the random walk system, the coefficients of the closed form of the Lainiotis filter are related to the Fibonacci numbers and the relation between the Lainiotis filter and the Fibonacci sequence has been established by (21) and (22).

Remark 5.1. Notice that the estimation error covariance converges to the steady state estimation error covariance irrespective of the initial condition $P(0/0)=P_0$. For every $k\geq 0$ from (21) we have

$$
P(k+1/k+1)=\frac{f_{2k+3}\sigma^2+f_{2k+2}P_0}{f_{2k+4}\sigma^2+f_{2k+3}P_0}\sigma^2=\frac{f_{2k+3}}{f_{2k+4}}\frac{\sigma^2+\dfrac{f_{2k+2}}{f_{2k+3}}P_0}{\sigma^2+\dfrac{f_{2k+3}}{f_{2k+4}}P_0}\sigma^2
$$

It is obvious that by (12) and (7) the above equality yields

$$
P_e=\lim_{k\to\infty}P(k/k)=\alpha\frac{\sigma^2+\alpha P_0}{\sigma^2+\alpha P_0}\sigma^2=\alpha\sigma^2
$$

5.3. *Steady state Lainiotis filter and the golden section*

The steady state Lainiotis filter for the random walk system in (14) with the process and measurement noise sources have noise covariances as in (15) takes advantage of the a priori knowledge of the steady state estimation covariance P_e by (20). Then, substituting the estimation covariance $P(k/k)$ by the steady state estimation covariance $P_e=\alpha\sigma^2$ in (18) we have

$$
\begin{aligned}
x(k+1/k+1) &= \frac{1}{2\sigma^2+\alpha\sigma^2}\sigma^2x(k/k)+\frac{\sigma^2+\alpha\sigma^2}{2\sigma^2+\alpha\sigma^2}z(k+1) \\
&= \frac{1}{2+\alpha}x(k/k)+\frac{1+\alpha}{2+\alpha}z(k+1)
\end{aligned}\tag{23}
$$

Using $1+\alpha=1/\alpha$ from (5), we derive

$$
\frac{1+\alpha}{2+\alpha}=\frac{1+\alpha}{1+1+\alpha}=\frac{1/\alpha}{1+(1/\alpha)}=\frac{1/\alpha}{(\alpha+1)/\alpha}=\frac{1}{\alpha+1}=\alpha
$$

and

$$
\frac{1}{2+\alpha}=\frac{1}{1+1+\alpha}=\frac{1}{1+(1/\alpha)}=\frac{\alpha}{\alpha+1}=\alpha^2
$$

Substituting the above quantities in (23) we derive the *recursive form* of the *steady state Lainiotis filter*

$$
x(k+1/k+1)=\alpha^2x(k/k)+\alpha z(k+1)\tag{24}
$$

for $k=0,1,\ldots$, with initial condition $x(0/0)=x_0$.

Thus, it becomes evident that the recursive form of the steady state Lainiotis filter computes the state estimate using a linear combination of *the previous estimate* and of *the current measurement* with coefficients related to the golden section.

Furthermore, for $k=0,1,\ldots$, and $x(0/0)=x_0$, the *closed form* of the *steady state Lainiotis filter* can be derived:

$$
x(k+1/k+1)=\alpha^{2(k+1)}x(0/0)+\sum_{i=1}^{k+1}\alpha^{2(k-i)+3}z(i)\tag{25}
$$

Proof. The proof of (25) is based on the induction method.

In fact, for $k=0$ we are able to write the recursive form of the steady state Lainiotis filter in (24) as

$$
x(1/1)=\alpha^2x(0/0)+\alpha z(1)
$$

which satisfies (25).

Assume that the formula in (25) is true for k, then by (24) and the assumption of induction we derive

$$
\begin{aligned}
x(k+2/k+2) &= \alpha^2x(k+1/k+1)+\alpha z(k+2) \\
&= \alpha^2\left[\alpha^{2(k+1)}x(0/0)+\sum_{i=1}^{k+1}\alpha^{2(k-i)+3}z(i)\right]+\alpha z(k+2) \\
&= \alpha^{2(k+2)}x(0/0)+\sum_{i=1}^{k+1}\alpha^{2(k-i)+5}z(i)+\alpha z(k+2) \\
&= \alpha^{2(k+2)}x(0/0)+\sum_{i=1}^{k+2}\alpha^{2(k-i)+5}z(i)
\end{aligned}
$$

i.e. (25) holds also for $k+1$, and hence for all $k\geq 0$. □

It becomes obvious that the steady state Lainiotis filter computes the state estimate as a linear combination of *the initial state estimate* and of *all previous measurements* with coefficients, which are powers of the golden section.

5.4. *FIR steady state Lainiotis filter and the golden section*

Using the ideas in [2] for the scalar stochastic dynamic system in (14) with $q=r=\sigma^2$ we are able to derive a *FIR* *form* of the *steady state Lainiotis filter* as

$$x(k/k) = \sum_{i=1}^{N} \alpha^{2(N-i)+1} z(i+k-N) \tag{26}$$

for $k > N$, where N such that

$$\alpha^{2N} < \varepsilon \tag{27}$$

and ε a small positive number.

Proof. Using the closed form of the steady state Lainiotis filter in (25) for $k > N$ we derive

$$\begin{aligned} x(k/k) &= \alpha^{2k} x(0/0) + \sum_{j=1}^{k} \alpha^{2(k-j)+1} z(j) \\ &= \alpha^{2k} x(0/0) + \sum_{j=1}^{k-N} \alpha^{2(k-j)+1} z(j) + \sum_{j=k-N+1}^{k} \alpha^{2(k-j)+1} z(j) \\ &= \alpha^{2k} x(0/0) + \sum_{j=1}^{k-N} \alpha^{2(k-j)+1} z(j) + \sum_{i=1}^{N} \alpha^{2(N-i)+1} z(i+k-N) \end{aligned} \tag{28}$$

Note that the golden section has the property $\lim_{N\to\infty} \alpha^{2N} = 0$ due to $\alpha < 1$. Consequently, by (27) there exists an integer N such that $\alpha^{2N} < \varepsilon$, which means that we are able to assume that $\alpha^j = 0$, for every $j \ge 2N$, while $\alpha^j \ne 0$, $j < 2N$.

The last property allows us to confirm that in (28) the coefficient of $x(0/0)$ tends to zero, since $2k > 2N$, and all the coefficients of $z(j)$ of the first sum for $1 \le j \le k-N$ tend to zero, since $2(k-j)+1 \ge 2N+1 > 2N$. Thus, it is obvious that (28) yields (26). □

The proposed FIR implementation of the steady state Lainiotis filter computes the state estimate as a linear combination of a *known number of the last measurements*, which are powers of the golden section α.

Table 1 summarizes the computational requirements required for the computation of the estimate value $x(k/k)$ of the state variable at time k of all the Lainiotis filter algorithms for the random walk system presented above.

An essential advantage of the FIR form of the steady state Lainiotis filter is that the calculation burden does not depend on the estimation time, leading to the reduction of the computational time in comparison to the other algorithms. The coefficients of the FIR form are powers of the golden section and are computed off-line. Recall that only N last measurements are required. The calculation burden depends on the length N. In fact, a small value of N is enough to give reliable estimates. This is confirmed through the experiment where a random walk system has been considered. Fig. 1 depicts the estimates $x(k/k)$ computed using the recursive form of the steady state Lainiotis filter with initial condition $x(0/0) = 0$ and the FIR form of the steady state Lainiotis filter for $N=5$. This choice is rational ($\alpha^{10} = 0.008131$ is of the order of 10^{-3}). It is noticeable that the estimates are close to each other (almost equivalent).

Table 1
Computational requirements of Lainiotis filter algorithms for the random walk system.

Lainiotis filter (LF) Algorithm	Equations	Calculation burden (scalar operations)	Order
Recursive form LF	(17) and (18)	$8k$	$8k$
Closed form LF	(21) and (22)	$7k+6$	$7k$
Recursive form steady state LF	(24)	$3k$	$3k$
Closed form steady state LF	(25)	$3k+1$	$3k$
FIR form steady state LF	(26)	$2N-1$	$2N$

6. Lainiotis filter, Kalman filter and the golden section

In this section the relation between the Lainiotis filter [3,9,10], the Kalman filter [1,8] and the golden section is described for the random walk system in (14) with $q = r = \sigma^2$ as in (15).

It is known that the Lainiotis filter is equivalent to the Kalman filter [3]. Thus, for the random walk system both filters provide the same estimates and the same estimation error covariances. Of course they provide the same steady state estimation error covariances as well.

From the Lainiotis filter equations the recursive Riccati equation for the estimation error covariance, $P(k/k)$, formulates in (17): $P(k+1/k+1) = ((\sigma^2 + P(k/k))/(2\sigma^2 + P(k/k)))\sigma^2$.

The relation between the smoothing error covariance, $P(k/k+1)$, and the estimation error covariance is obtained by the Lainiotis filter equation $P(k/k+1) = [I + P(k/k)O_n]^{-1} P(k/k)$, where substituting O_n by (16) arises:

$$P(k/k+1) = \frac{2P(k/k)}{2\sigma^2 + P(k/k)} \sigma^2 \tag{29}$$

Since Eq. (29) yields

$$P(k/k) = \frac{2P(k/k+1)}{2\sigma^2 - P(k/k+1)} \sigma^2 \tag{30}$$

combining (17) and (30) we are able to derive the recursive Riccati equation for the smoothing error covariance from (29) as follows:

$$\begin{aligned} P(k/k+1) &= \frac{2\sigma^2 \dfrac{\sigma^2 + P(k-1/k-1)}{2\sigma^2 + P(k-1/k-1)} \sigma^2}{2\sigma^2 + \dfrac{\sigma^2 + P(k-1/k-1)}{2\sigma^2 + P(k-1/k-1)} \sigma^2} \\ &= 2\sigma^2 \frac{\sigma^2 + P(k-1/k-1)}{5\sigma^2 + 3P(k-1/k-1)} \\ &= 2\sigma^2 \frac{\sigma^2 + \dfrac{2P(k-1/k)}{2\sigma^2 - P(k-1/k)} \sigma^2}{5\sigma^2 + 3\dfrac{2P(k-1/k)}{2\sigma^2 - P(k-1/k)} \sigma^2} \\ &= 2\sigma^2 \frac{2\sigma^2 + P(k-1/k)}{10\sigma^2 + P(k-1/k)} \end{aligned}$$

Thus, the recursive Riccati equation for the smoothing error covariance is given by

$$P(k/k+1) = \frac{4\sigma^2 + 2P(k-1/k)}{10\sigma^2 + P(k-1/k)} \sigma^2 \tag{31}$$

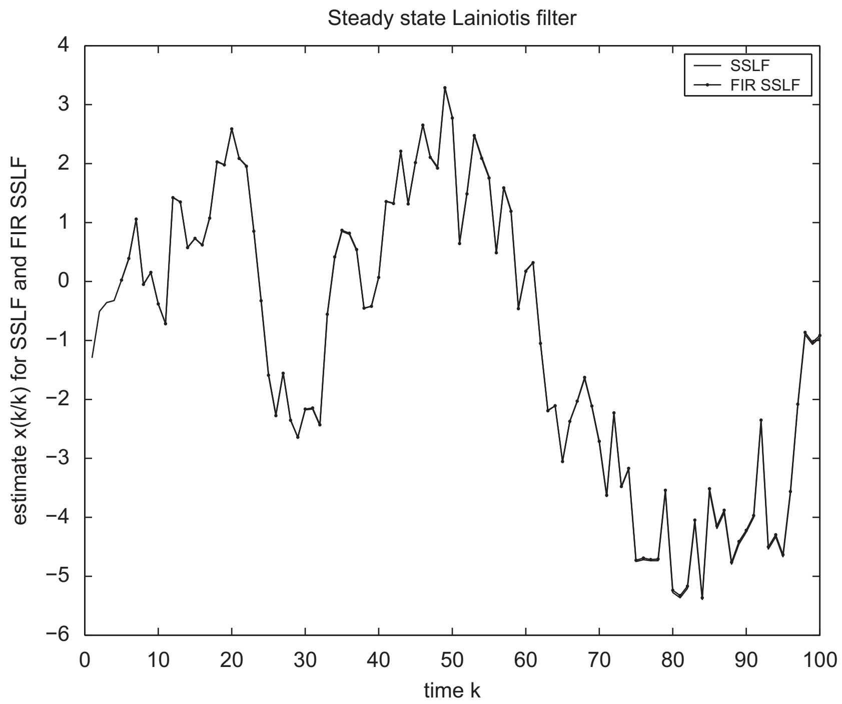

Fig. 1. Estimates computed using the recursive form of the steady state Lainiotis filter and the FIR form of the steady state Lainiotis filter for $N=5$.

The relation between the prediction error covariance $P(k+1/k)$, and the estimation error covariance, is obtained by the equation $P(k+1/k)=FP(k/k)F^T+Q$ of the Kalman filter [3], where substituting $F=f=1$, $Q=q=\sigma^2$ arises:

$$P(k+1/k)=P(k/k)+\sigma^2 \tag{32}$$

Combining (17) and (32) we are able to derive the recursive Riccati equation for the prediction error covariance:

$$P(k+1/k)=\frac{\sigma^2+2P(k/k-1)}{\sigma^2+P(k/k-1)}\sigma^2 \tag{33}$$

By (29) and (32) it is easy to conclude that

$$P(k/k+1)<P(k/k)<P(k+1/k) \tag{34}$$

The relations in (34) mean that the smoothed value of the state is better that the estimated one and that the estimated value of the state is better that the predicted one. The smoothing/estimation/prediction error covariances tend to the corresponding steady state smoothing/estimation/prediction error covariances $P_s/P_e/P_p$, respectively. Of course, from the recursive Riccati equations in (17), (31) and (33) and using the relations in (3) and (5) we are able to derive the corresponding algebraic Riccati equations and their steady state solutions, which are summarized in Table 2:

$$P_s=\lim_{k\to\infty}P(k/k+1)=2\alpha^3\sigma^2$$

$$P_e=\lim_{k\to\infty}P(k/k)=\alpha\sigma^2$$

$$P_p=\lim_{k\to\infty}P(k+1/k)=\frac{1}{\alpha}\sigma^2$$

Table 2
Steady state error covariances and golden section.

Error covariance	Algebraic Riccati equation	Steady state error covariance
Smoothing	$P_s^2+8\sigma^2P_s-4\sigma^4=0$	$P_s=2\alpha^3\sigma^2$
Estimation	$P_e^2+\sigma^2P_e-\sigma^4=0$	$P_e=\alpha\sigma^2$
Prediction	$P_p^2-\sigma^2P_p-\sigma^4=0$	$P_p=\frac{1}{\alpha}\sigma^2$

By the three above equalities it is easy to conclude that

$$P_s<P_e<P_p$$

It is clear that in the special case $q=r=\sigma^2=1$, we have

$$P_s=2\alpha^3<P_e=\alpha<P_p=\frac{1}{\alpha}$$

It is apparent that for the random walk system in (14) and (15), the steady state smoothing/estimation/prediction error covariances emanating from Lainiotis and Kalman filters are related to the golden section, as depicted in Table 2.

7. Lainiotis filter for the scalar generic stochastic dynamic system

Consider the scalar generic stochastic dynamic system in (8) with the transition coefficient $F=f$ and output $H=h$

$$x(k+1)=fx(k)+w(k)$$

$$z(k+1)=hx(k+1)+v(k+1)$$

for $k=0,1,\ldots$, and the process and measurement noise sequences with covariances given by $Q=q$ and $R=r$, respectively, with $q>0$ and $r>0$. Then the Lainiotis filter parameters by (11) are

$$A=\frac{1}{qh^2+r},\quad K_n=\frac{qh}{qh^2+r},\quad K_m=\frac{fh}{qh^2+r},$$
$$P_n=\frac{qr}{qh^2+r},\quad F_n=\frac{rf}{qh^2+r},\quad O_n=\frac{f^2h^2}{qh^2+r} \tag{35}$$

The aim of this section is to investigate for which values of the parameters f,h,q,r the steady state Lainiotis filter is related to the golden section α.

Substituting the values of the parameters by (35) in (9) and (10), the *recursive form* of the *Lainiotis filter* is derived:

$$P(k+1/k+1)=\frac{qr}{qh^2+r}+\frac{r^2f^2}{(qh^2+r)(qh^2+r+f^2h^2P(k/k))}P(k/k) \tag{36}$$

$$x(k+1/k+1)=\frac{rf}{qh^2+r+f^2h^2P(k/k)}x(k/k)+\left(\frac{qh}{qh^2+r}+\frac{rf^2hP(k/k)}{(qh^2+r)(qh^2+r+f^2h^2P(k/k))}\right)z(k+1) \tag{37}$$

for $k=0,1,\ldots$, with initial conditions $P(0/0)=P_0$, and $x(0/0)=x_0$.

Note that there always exists a unique positive steady state value P_e of the estimation error covariance, i.e. the estimation error covariance $P(k/k)$ tends to the steady state estimation error covariance P_e, which can be calculated by recursively implementing the Riccati equation emanating from Lainiotis filter (36) with initial condition $P(0/0)=P_0$. Furthermore, the steady state value P_e of the estimation error covariance satisfies the corresponding algebraic Riccati equation:

$$f^2h^2P_e^2+(qh^2+r-rf^2)P_e-qr=0 \tag{38}$$

Notice that the discriminant $\Delta=(qh^2+r-rf^2)^2+4qrf^2h^2$ is a positive number, thus the real roots of (38) exist always and its unique positive root is given by

$$P_e=\frac{-(qh^2+r-rf^2)+\sqrt{(qh^2+r-rf^2)^2+4qrf^2h^2}}{2f^2h^2} \tag{39}$$

The steady state Lainiotis filter for the scalar generic stochastic dynamic system takes advantage of the a priori knowledge of the steady state estimation covariance P_e by (39). Then, substituting the estimation covariance $P(k/k)$ by the steady state estimation covariance P_e in Eq. (37), the *recursive form* of the *steady state Lainiotis filter* is derived:

$$x(k+1/k+1)=\frac{rf}{qh^2+r+f^2h^2P_e}x(k/k)+\left(\frac{qh}{qh^2+r}+\frac{rf^2hP_e}{(qh^2+r)(qh^2+r+f^2h^2P_e)}\right)z(k+1) \tag{40}$$

for $k=0,1,\ldots$, with initial condition $x(0/0)=x_0$.

The relation between the steady state estimation error covariance P_e, the parameters f,h,q,r and the golden section α is presented in the following supposing that holds:

$$0<\alpha\frac{q}{r}h^2<1 \tag{41}$$

In fact, the following relations are equivalent:

$$\text{(i)}\ P_e=\frac{r}{h^2}\alpha \tag{42}$$

$$\text{(ii)}\ f^2=\frac{r-\alpha qh^2}{r\alpha^2} \tag{43}$$

Proof. (i) $\Rightarrow$ (ii) Equating the forms of P_e by (39) and (42) we take

$$\sqrt{(qh^2+r-rf^2)^2+4qrf^2h^2}=2\alpha rf^2+qh^2+r-rf^2 \tag{44}$$

The right part of (44) is positive, since it is written

$(2\alpha-1)rf^2+qh^2+r=\alpha^3rf^2+qh^2+r$

due to $2\alpha-1=\alpha-\alpha^2=\alpha(1-\alpha)=\alpha^3$. Hence, the form of f^2 in (43) is derived directly by (44). It is obvious that $f^2>0$ due to inequality in (41).

(ii) $\Rightarrow$ (i) Substituting f^2 by (43) in (39) the estimation error covariance P_e in (42) directly arises. □

Thus, it has been shown that for the scalar generic stochastic dynamic system, the steady state estimation error covariance P_e is related to the golden section, under the assumption that the parameters f,h,q,r are related to the golden section α with the relation in (43).

In the special case where the inequality in (41) holds, substituting the quantities P_e,f^2 by (42) and (43) in (40) and using (5), the *special recursive form* of the *steady state Lainiotis filter* is derived:

$$x(k+1/k+1)=\alpha^2fx(k/k)+\frac{\alpha}{h}z(k+1) \tag{45}$$

for $k=0,1,\ldots$, with initial condition $x(0/0)=x_0$.

Thus, it has been shown that for the scalar generic stochastic dynamic system, the corresponding steady state Lainiotis filter is related to the golden section, under the assumption that the parameters f,h,q,r are related to the golden section α with the relation in (45).

Remark 7.1. It is clear that the Lainiotis filter as well as the steady state Lainiotis filter for the scalar stochastic dynamic system presented in Section 5 is verified. In fact, for the values of parameters $f=h=1$ and $q=r=\sigma^2$ the recursive form of the Lainiotis filter in (36) and (37) takes the form in (17) and (18).

Obviously, the choice of parameters $h=1$ and $q=r=\sigma^2$ satisfies the inequality in (41) and gives $P_e=\alpha\sigma^2$, as in (20); in this case, from (43) arises $f^2=1$ and choosing $f=1$ the special recursive form of the steady state Lainiotis filter in (45) takes the form in (24).

Furthermore, in the special case where the inequality in (41) and the equivalent relations in (42) and (43) hold, the *special closed form* of the *steady state Lainiotis filter* is derived for $k=0,1,\ldots$

$$x(k+1/k+1)=(\alpha^2 f)^{k+1}x(0/0)+\frac{\alpha}{h}\sum_{i=1}^{k+1}(\alpha^2 f)^{k-i+1}z(i) \tag{46}$$

Proof. It is easy to see that for $k=0$ the special recursive form in (45) is written as

$$x(1/1)=\alpha^2 f x(0/0)+\frac{\alpha}{h}z(1),$$

which satisfies (46).

By induction, supposing that Eq. (46) holds for k, then by (45) we derive

$$\begin{aligned}x(k+2/k+2)&=(\alpha^2 f)x(k+1/k+1)+\frac{\alpha}{h}z(k+2)\\&=(\alpha^2 f)\left[(\alpha^2 f)^{k+1}x(0/0)+\frac{\alpha}{h}\sum_{i=1}^{k+1}(\alpha^2 f)^{k-i+1}z(i)\right]+\frac{\alpha}{h}z(k+2)\\&=(\alpha^2 f)^{k+2}x(0/0)+\frac{\alpha}{h}\sum_{i=1}^{k+2}(\alpha^2 f)^{(k+1)-i+1}z(i)\end{aligned}$$

i.e. (46) holds also for $k+1$, and hence for all $k\geq 0$. □

It becomes obvious that, when the inequality in (41) and one of two equivalent relations in (42) and (43) hold, the steady state Lainiotis filter computes the state estimate as a linear combination of *the initial state estimate* and of *all previous measurements* with coefficients, which are related to the golden section α.

Finally, following the methodology used for the derivation of the FIR steady state Lainiotis filter in Section 5, in the special case where the inequality in (41) and the equivalent relations in (42) and (43) hold, we are able to derive the *special FIR form* of the *steady state Lainiotis filter* for the scalar generic stochastic dynamic system as

$$x(k/k)=\frac{\alpha}{h}\sum_{i=1}^{N}(\alpha^2 f)^{N-i}z(k-N+i) \tag{47}$$

for $k>N$, where N such that

$$(\alpha^2 f)^N<\varepsilon \tag{48}$$

and ε a small positive number.

Proof. Using the special closed form of the steady state Lainiotis filter in (46) for $k>N$ we derive:

$$\begin{aligned}x(k/k)&=(\alpha^2 f)^k x(0/0)+\frac{\alpha}{h}\sum_{j=1}^{k}(\alpha^2 f)^{k-j}z(j)\\&=(\alpha^2 f)^k x(0/0)+\frac{\alpha}{h}\sum_{j=1}^{k-N}(\alpha^2 f)^{k-j}z(j)+\frac{\alpha}{h}\sum_{j=k-N+1}^{k}(\alpha^2 f)^{k-j}z(j)\\&=(\alpha^2 f)^k x(0/0)+\frac{\alpha}{h}\sum_{j=1}^{k-N}(\alpha^2 f)^{k-j}z(j)+\frac{\alpha}{h}\sum_{i=1}^{N}(\alpha^2 f)^{N-i}z(k-N+i)\end{aligned} \tag{49}$$

Using the assumption $0<\alpha(q/r)h^2<1$ by (41) and the relation $f^2=(r-\alpha qh^2)/r\alpha^2$, it becomes clear that the quantity $\alpha^2 f$ has the property $\lim_{N\to\infty}(\alpha^2 f)^N=0$ due to

$$|\alpha^2 f|=\alpha^2\sqrt{\frac{r-\alpha qh^2}{r\alpha^2}}=\alpha\sqrt{1-\frac{\alpha qh^2}{r}}<1.$$

Consequently, by (48) there exists an integer N such that $(\alpha^2 f)^N<\varepsilon$, which means that we are able to assume that $(\alpha^2 f)^j=0$, for every $j\geq N$, while $(\alpha^2 f)^j\neq 0$, $j<N$. The last property allows us to confirm that in (49) the coefficient of $x(0/0)$ tends to zero, since $k>N$, and all the coefficients of $z(j)$ of the first sum for $1\leq j\leq k-N$ tend to zero, since $k-j\geq N$. Thus, it is obvious that (49) yields (47). □

In the case where the inequality in (41) and the equivalent relations in (42) and (43) hold, the special FIR implementation of the steady state Lainiotis filter computes the state estimate as a linear combination of a *known number of the last measurements* with coefficients, which are powers of the golden section α.

8. Conclusions

The relation between the discrete time Lainiotis filter on the one side and the golden section and the Fibonacci sequence on the other side is established.

Consider the random walk system, i.e. the scalar stochastic dynamic system with the transition and output coefficients equal to one. It is shown that the Lainiotis filter computes the state estimate using a linear combination of the previous estimate and of the current measurement with coefficients related to the Fibonacci numbers. Furthermore, it is pointed out that the steady state estimation error covariance is related to the golden section. It is also shown that the recursive form of the steady state Lainiotis filter computes the state estimate using a linear combination of the previous estimate and the current measurement with coefficients related to the golden section, while the non-recursive form of the steady state Lainiotis filter computes the state estimate as a linear combination of the initial state estimate and of all previous measurements with powers of the golden section as coefficients.

A FIR implementation of the steady state Lainiotis filter is also proposed, where the filter computes the state estimate as a linear combination of a well-defined set of the last measurements with coefficients which are powers of the golden section.

Table 3 summarizes the relationship between the Lainiotis filter, the golden section and the Fibonacci numbers for the scalar stochastic dynamic system. Thus, it becomes evident that for the scalar stochastic dynamic

Table 3
Relationship between the Lainiotis filter, the golden section and the Fibonacci numbers.

Lainiotis filter algorithm	Relation to
Closed form Lainiotis filter	Fibonacci numbers
Recursive form Lainiotis filter	Golden section
Recursive form steady state Lainiotis filter	Golden section
Closed form steady state Lainiotis filter	Golden section
FIR steady state Lainiotis filter	Golden section

system the Lainiotis filter is fully governed by the golden section and the Fibonacci sequence.

Concerning to the scalar generic stochastic dynamic system the relation between the parameters and the golden section was investigated. Results analogous to those for the scalar stochastic dynamic system were derived under the assumption that the parameters are related to the golden section with a specified relation.

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